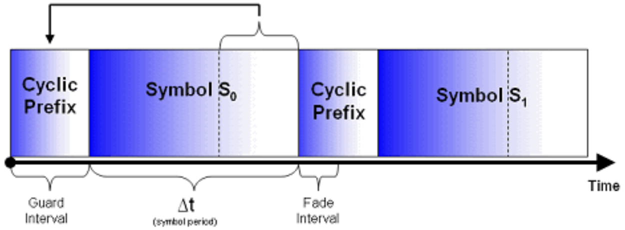

ECS455: Chapter 5

OFDM

5.4 Cyclic Prefix (CP)

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Tuesday	**9:30-10:30**
Tuesday	**13:30-14:30**
Thursday	**13:30-14:30**

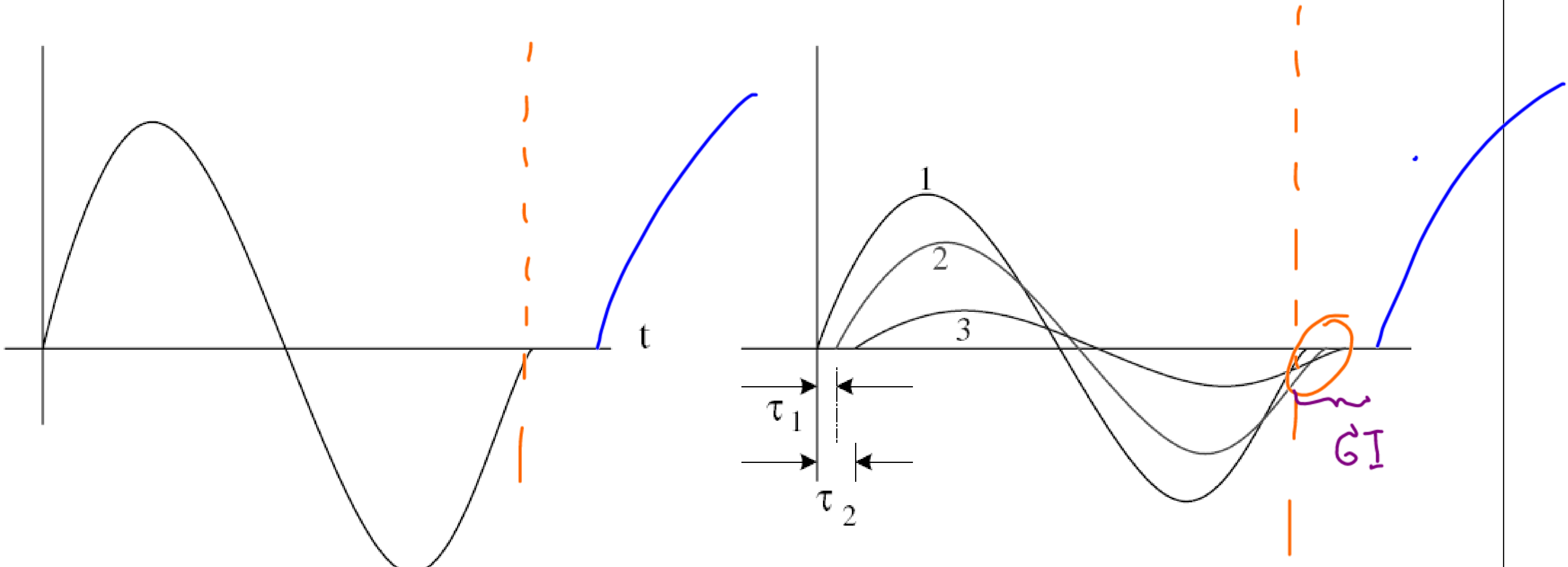
Three steps towards modern OFDM

1. Mitigate Multipath (ISI): Decrease the rate of the original data stream via multicarrier modulation (FDM)
2. Gain Spectral Efficiency: Utilize orthogonality
3. Achieve Efficient Implementation: FFT and IFFT

- Extra step: Completely eliminate ISI and ICI
  - Cyclic prefix

Cyclic Prefix: Motivation (1)

- Recall: Multipath Fading and Delay Spread



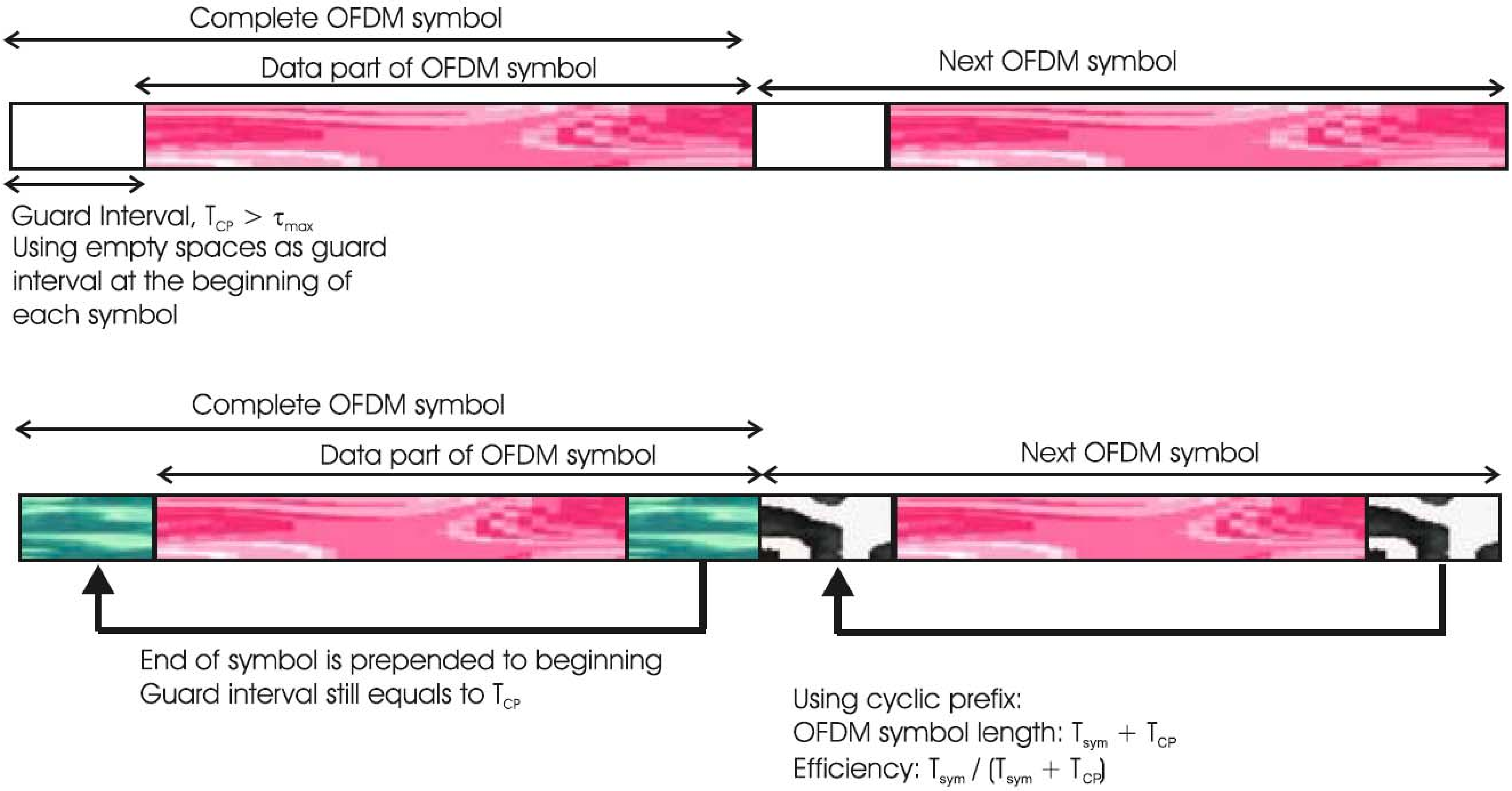
Transmitted Signal

Received Signal

Cyclic Prefix: Motivation (2)

- OFDM uses large symbol duration T_s
  - compared to the duration of the impulse response $\tau_{\max}$ of the channel
  - to reduce the amount of ISI
- **Q**: Can we "eliminate" the multipath (**ISI**) problem?
- **A**: To reduce the ISI, add **guard interval** larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., **ICI** (inter-channel interference) still exists.
- **Solution**: To prevent **both** the **ISI** as well as the **ICI**, OFDM symbol is **cyclically extended** into the guard interval.

Cyclic Prefix



Complete OFDM symbol

Data part of OFDM symbol

Next OFDM symbol

Guard Interval, $T_{CP} > \tau_{max}$
Using empty spaces as guard interval at the beginning of each symbol

Complete OFDM symbol

Data part of OFDM symbol

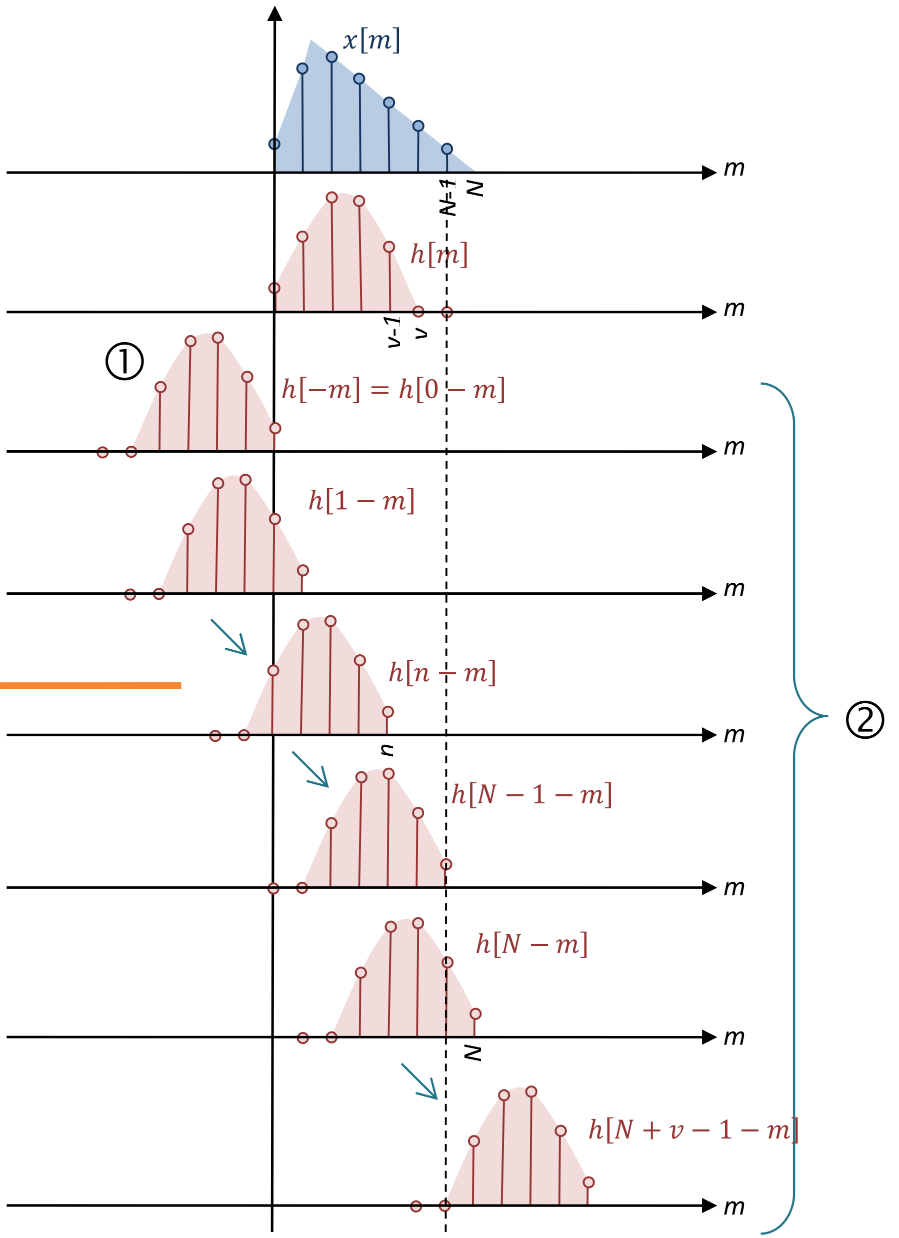
Next OFDM symbol

End of symbol is prepended to beginning
Guard interval still equals to T_{CP}

Using cyclic prefix:
OFDM symbol length: $T_{sym} + T_{CP}$
Efficiency: $T_{sym} / (T_{sym} + T_{CP})$

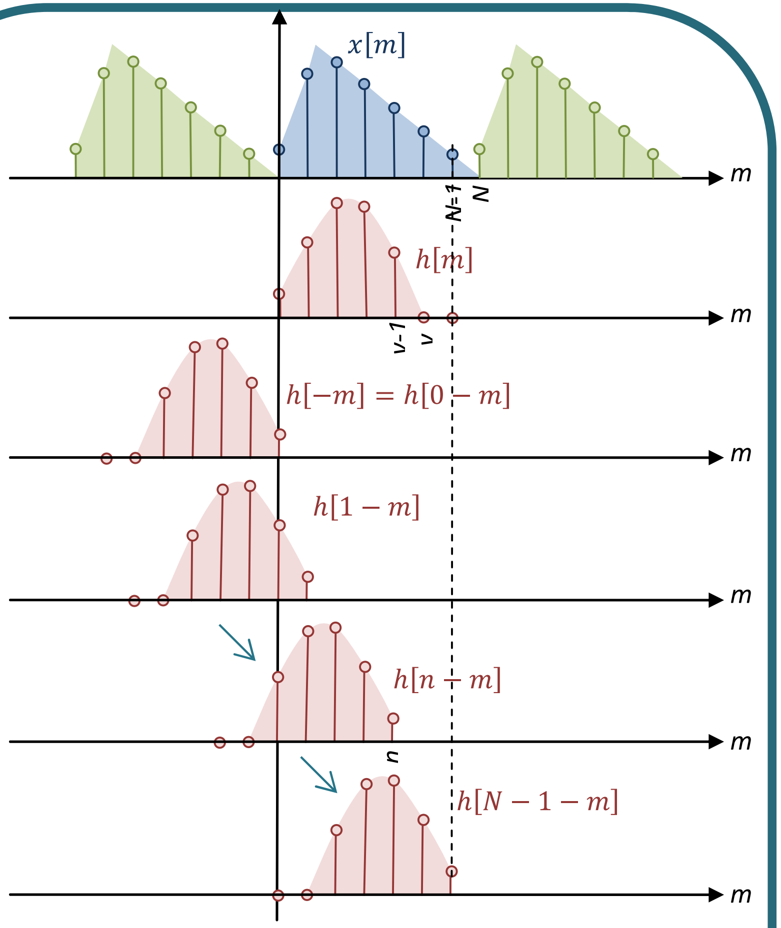
Recall: Convolution

① Flip
② Shift
③ Multiply (pointwise)
④ Add



$$\{x*h\}[n]=\sum_{m}x[m]h[n-m]$$

Circular Convolution

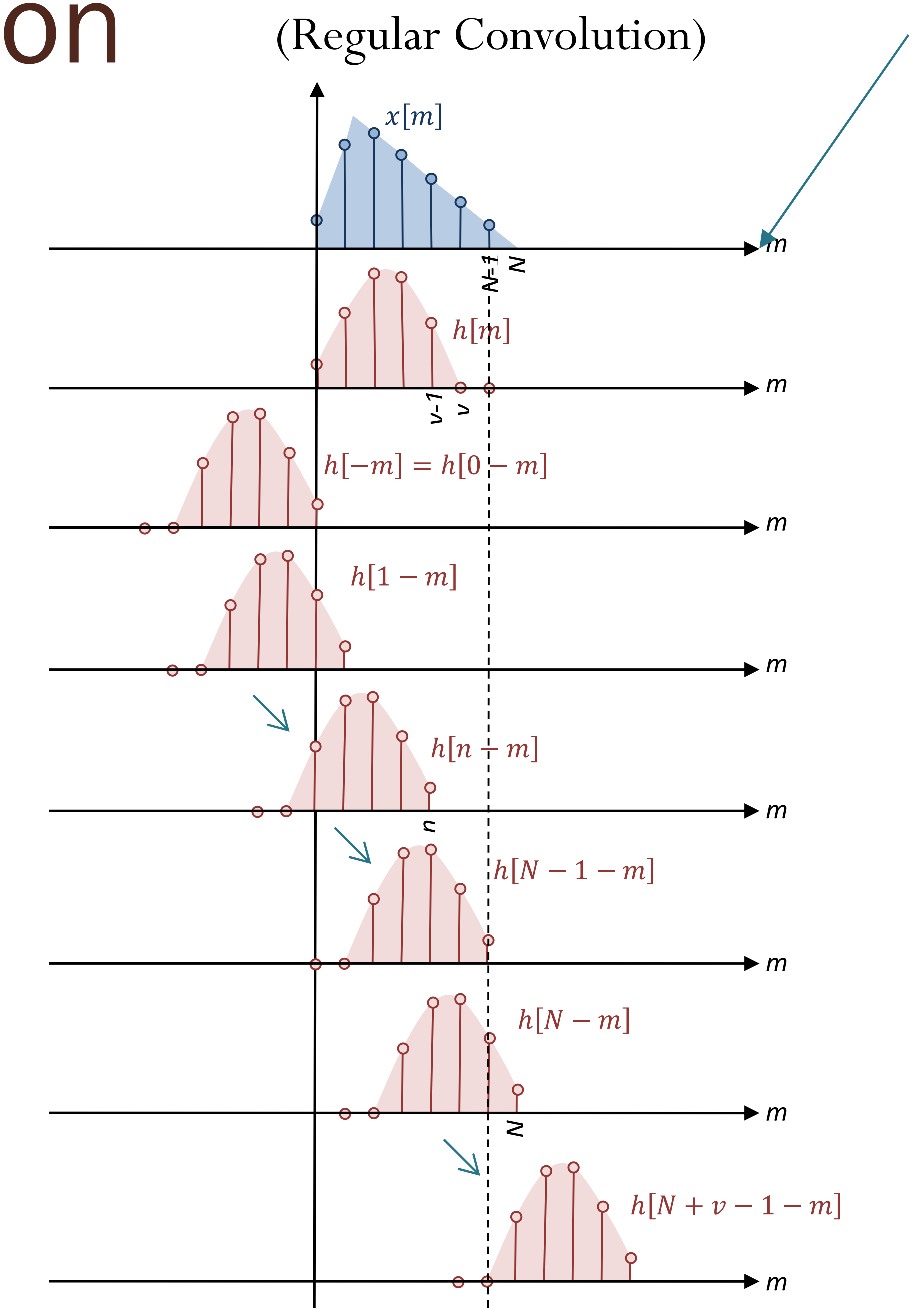


⓪ Replicate *x* (now it looks periodic)

Then, perform the usual convolution

⑤ only on n = 0 to *N*-1

(Regular Convolution)



Circular Convolution: Examples 1

Find

$$[1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

```
>> conv([1,2,3],[4,5,6])
ans =
       4      13      28      27      18
```

$$[1 \quad 2 \quad 3] \circledast [4 \quad 5 \quad 6]$$

```
>> cconv([1,2,3],[4,5,6],3)
ans =
     31      31      28
```

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0]$$

```
>> cconv([1,2,3,0,0],[4,5,6,0,0],5)
ans =
    4.0000    13.0000    28.0000    27.0000    18.0000
```

① 1 2 3
6 5 4 — 4
6 5 4 — 5+8 = 13
6 5 4 — 6+10+12 = 28
② 6 5 4 — 12+15 = 27
6 5 4 — 18

⓪ 1 2 3 1 2 3 1 2 3
6 5 4 — 12+15+4 = 31
6 5 4 — 18+5+8 = 31
6 5 4 — 6+10+12 = 28

⑤

Discussion

- ***Regular convolution*** of an N_1–point vector and an N_2–point vector gives (N_1+N_2-1)-point vector.
- ***Circular convolution*** is performed between two equal-length vectors. The results also has the same length.
- Circular convolution can be used to find regular convolution by **zero-padding**.
  - Zero-pad the vectors so that their length is N_1+N_2-1.
  - Example:

$$\begin{bmatrix}1 & 2 & 3 & 0 & 0\end{bmatrix} \circledast \begin{bmatrix}4 & 5 & 6 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\end{bmatrix} * \begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

- In modern OFDM, we want to perform circular convolution via regular convolution.

Circular Convolution in Communication

- We want the receiver to obtain the circular convolution of the signal (channel input) and the channel.
- Q: Why?
- A:
  - **CTFT**: **convolution** in time domain corresponds to **multiplication** in frequency domain.
    - This fact does not hold for DFT.
  - **DFT**: circular **convolution** in (discrete) time domain corresponds to **multiplication** in (discrete) frequency domain.
    - We want to have multiplication in frequency domain.
    - So, we want circular convolution and not the regular convolution.
- Problem: Real channel does regular convolution.
- Solution: With **cyclic prefix**, regular convolution can be used to create circular convolution.

Example 2

$$\underset{x}{[1 \quad -2 \quad 3 \quad 1 \quad 2]} \circledast [\underbrace{3 \quad 2 \quad 1}_{h} \quad \overbrace{0 \quad 0}] = ?$$

Solution:

Let's look closer at how we carry out the circular convolution operation. Recall that we replicate the *x* and then perform the regular convolution (for *N* points)

1	–2	3	1	2	1	–2	3	1	2	1	–2	3	1	2	
	0	0	1	2	3										$1\times1+2\times2+1\times3=1+4+3=8$
		0	0	1	2	3									$2\times1+1\times2+(-2)\times3=2+2-6=-2$
			0	0	1	2	3								$1\times1+(-2)\times2+3\times3=1-4+9=6$
				0	0	1	2	3							$(-2)\times1+3\times2+1\times3=-2+6+3=7$
					0	0	1	2	3						$3\times1+1\times2+2\times3=3+2+6=11$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = \underbrace{[8 \quad -2 \quad 6 \quad 7 \quad 11]}$$

Goal: Get these numbers using regular convolution

Example 2

$$[1\ \ -2\ \ 3\ \ 1\ \ 2] \circledast [\overbrace{\underset{h[0]}{3}\ \ \underset{h[1]}{2}\ \ \overset{\overset{2}{h[2]}}{1}}^{h}\ \ 0\ \ 0] = ?$$

Observation: We don't need to replicate the *x* indefinitely. Furthermore, when *h* is shorter than *x*, we need only a part of one replica.

Not needed in the calculation

1	−2	3	1	2	1	−2	3	1	2	1	−2	3	1	2	
	0	0	1	2	3										$1\times1+2\times2+1\times3=1+4+3=8$
		0	0	1	2	3									$2\times1+1\times2+(-2)\times3=2+2-6=-2$
			0	0	1	2	3								$1\times1+(-2)\times2+3\times3=1-4+9=6$
				0	0	1	2	3							$(-2)\times1+3\times2+1\times3=-2+6+3=7$
					0	0	1	2	3						$3\times1+1\times2+2\times3=3+2+6=11$

$$[1\ \ -2\ \ 3\ \ 1\ \ 2] \circledast [3\ \ 2\ \ 1\ \ 0\ \ 0] = [8\ \ -2\ \ 6\ \ 7\ \ 11]$$

Example 2

Try this: use only the necessary part of the replica and then convolute (regular convolution) with the channel.

$$[\underbrace{1\quad 2}\quad 1\quad -2\quad 3\quad 1\quad 2]*[3\quad 2\quad 1]=?$$

Copy the **last ν samples** of the symbols at the **beginning** of the symbol.
This partial replica is called the **cyclic prefix**.

$$\overbrace{1\quad 2}\quad 1\quad -2\quad 3\quad 1\quad 2$$

1 2 3

1 2 3

1 2 3

1 2 3

1 2 3

1 2 3

1 2 3

1 2 3

1 2 3

$1\times 3=3$

$1\times 2+2\times 3=2+6=8$

$1\times 1+2\times 2+1\times 3=1+4+3=8$

$2\times 1+1\times 2+(-2)\times 3=2+2-6=-2$

$1\times 1+(-2)\times 2+3\times 3=1-4+9=6$

$(-2)\times 1+3\times 2+1\times 3=-2+6+3=7$

$3\times 1+1\times 2+2\times 3=3+2+6=11$

$1\times 1+2\times 2=1+4=5$

$2\times 1=2$

Junk!

Example 2

- We now know that

$$[\underbrace{1\quad 2}_{\text{Cyclic Prefix}}\quad 1\quad -2\quad 3\quad 1\quad 2]*[3\quad 2\quad 1]=[3\quad 8\quad \underbrace{8\quad -2\quad 6\quad 7\quad 11}_{[1\quad -2\quad 3\quad 1\quad 2]\circledast[3\quad 2\quad 1\quad 0\quad 0]}\quad 5\quad 2]$$

- Similarly, you may check that

$$[\underbrace{-2\quad 1}_{\text{Cyclic Prefix}}\quad 2\quad 1\quad -3\quad -2\quad 1]*[3\quad 2\quad 1]=[-6\quad -1\quad \underbrace{6\quad 8\quad -5\quad -11\quad -4}_{[2\quad 1\quad -3\quad -2\quad 1]\circledast[3\quad 2\quad 1\quad 0\quad 0]}\quad 0\quad 1]$$

Example 3

- We know, from Example 2, that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
```

And that

```
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Check that

```
  [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0] * [3 2 1]
= [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
```

and

```
  [ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] * [3 2 1]
= [ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]
```

Example 4

- We know that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Using Example 3, we have

```
   [ 1 2 1 -2 3  1  2 -2  1 2 1 -3  -2  1] * [3 2 1]
= ( [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0] )
  (+[ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] ) * [3 2 1]
=   [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
   +[ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]
=   [ 3 8 8 -2 6  7 11 -1  1 6 8 -5 -11 -4 0 1]
```

Putting results together…

- Suppose $\underline{x}^{(1)} = [1\ -2\ 3\ 1\ 2]$ and $\underline{x}^{(2)} = [2\ 1\ -3\ -2\ 1]$
- Suppose $\underline{h} = [3\ 2\ 1]$
- At the receiver, we want to get
  - $[1\ -2\ 3\ 1\ 2] \circledast [3\ 2\ 1\ 0\ 0] = [8\ -2\ 6\ 7\ 11]$
  - $[2\ 1\ -3\ -2\ 1] \circledast [3\ 2\ 1\ 0\ 0] = [6\ 8\ -5\ -11\ -4]$
- We transmit $[\ \underbrace{1\ 2}_{\text{Cyclic prefix}}\ 1\ -2\ 3\ \ 1\ \ 2\ \underbrace{-2\ \ 1}_{\text{Cyclic prefix}}\ 2\ 1\ -3\ \ -2\ \ 1]$.
- At the receiver, we get

$$[\ 1\ 2\ 1\ -2\ 3\ \ 1\ \ 2\ -2\ \ 1\ 2\ 1\ -3\ \ -2\ \ 1] * [3\ 2\ 1]$$

$$= [\ \underbrace{3\ 8}\ 8\ -2\ 6\ \ 7\ 11\ \underbrace{-1\ \ 1}\ 6\ 8\ -5\ -11\ -4\ \underbrace{0\ 1}]$$

Junk! To be thrown away by the receiver.

Circular Convolution: Key Properties

- Consider an N-point signal $x[n]$
- **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last ν samples of the symbols at the beginning of the symbol:

$$\hat{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ x[n+N], & -v \le n \le -1 \end{cases}$$

- Key Property 1:

$$\{h \circledast x\}[n] = (h * \hat{x})[n] \text{ for } 0 \le n \le N-1$$

- Key Property 2:

$$\{h \circledast x\}[n] \xrightarrow{\text{FFT}} \boxed{H_k X_k}$$

$$\mathbf{h} = \left(h[0], h[1], h[2], \ldots h[\nu]\right) \qquad \mathbf{H} = \mathrm{FFT}\left(\tilde{\mathbf{h}}\right)$$

zero-padded to length N

OFDM with CP for Channel w/ Memory

- To send N samples $\mathbf{S} = (S_0, S_1, \ldots, S_{N-1})$
- First apply IFFT with scaling by $\sqrt{N}$: $\tilde{\mathbf{s}} = \sqrt{N}\,\mathrm{IFFT}\left(\mathbf{S}\right)$
- Then, add cyclic prefix

$$\mathbf{x} = \left[\tilde{s}[N-\nu], \ldots, \tilde{s}[N-1], \tilde{s}[0], \ldots, \tilde{s}[N-1]\right]$$

- This is inputted to the channel
- The channel output is $\mathbf{y} = \mathbf{x} * \mathbf{h}$ which can be viewed as

$$\mathbf{y} = \left[p[N-\nu], \ldots, p[N-1], r[0], \ldots, r[N-1]\right]$$

- Remove cyclic prefix to get $\mathbf{r}$. (We know that $\mathbf{r} = \tilde{\mathbf{s}} \circledast \mathbf{h}$.)
- Then apply FFT with scaling by $^1/_{\sqrt{N}}$: $\tilde{\mathbf{R}} = \frac{1}{\sqrt{N}}\mathrm{FFT}\left(\mathbf{r}\right)$
- By circular convolution property of DFT,

$$\mathbf{r} = \tilde{\mathbf{s}} \circledast \mathbf{h} \Longrightarrow R_k = H_k \tilde{S}_k \Longrightarrow \tilde{R}_k = H_k S_k \Longrightarrow S_k = \frac{\tilde{R}_k}{H_k}$$

No ICI!

MATLAB Example (1/2)

```
S = [1 -1 2 4 5 -1 2 -3]; % data stream
h = [1 0.3 0.1];

% OFDM transmitter
N = 4;                            % Number of data symbols per OFDM symbol
n = length(S)/N;                  % Number of data blocks
St = (reshape(S,N,n)).';          % Reshape stream to matrix for
                                  % easier addition of cyclic prefix
st = (sqrt(N))*ifft(St,[],2);     % Calculate the IFFT with scaling
```

```
St =
      1     -1      2      4
      5     -1      2     -3
```

IFFT

w/ scaling

(row-wise)

```
st =
   3.0 + 0.0i   -0.5 - 2.5i   0.0 + 0.0i   -0.5 + 2.5i
   1.5 + 0.0i    1.5 + 1.0i   5.5 + 0.0i    1.5 - 1.0i
```

```
v = length(h)-1;
xt = [st(:,(N-(v-1)):N) st];      % Add Cyclic Prefix
x = (reshape(xt.',((N+v)*n),1)).'; % Reshape back to stream
```

```
x =
   0.0 + 0.0i  -0.5 + 2.5i   3.0 + 0.0i  -0.5 - 2.5i   0.0 + 0.0i   -0.5 + 2.5i   5.5 + 0.0i   1.5 - 1.0i   1.5 + 0.0i
1.5 + 1.0i   5.5 + 0.0i   1.5 - 1.0i
```

MATLAB Example (2/2)

```
%----------------------------
% Convolve with channel
y = conv(x,h);
H = fft([h zeros(1,N-v-1)]);
%----------------------------
% OFDM receiver
y = y(1:((N+v)*n));
yt = reshape(y,(N+v),n).';        % Reshape matrix for easier
                                   % removal of cyclic prefix
r = yt(:,v+1:v+N);                 % Eliminate junk (cyclic prefix)
Rt = (1/sqrt(N))*fft(r,[],2);      % Calculate the FFT with scaling
```

```
y =
   0.00 + 0.00i  -0.50 + 2.50i   2.85 + 0.75i   0.35 - 2.25i   0.15 - 0.75i
-0.55 + 2.25i   5.35 + 0.75i   3.10 - 0.75i   2.50 - 0.30i   2.10 + 0.90i
6.10 + 0.30i   3.30 - 0.90i
```

```
r =
   2.85 + 0.75i   0.35 - 2.25i   0.15 - 0.75i  -0.55 + 2.25i
   2.50 - 0.30i   2.10 + 0.90i   6.10 + 0.30i   3.30 - 0.90i
```

FFT

w/ scaling

(row-wise)

```
Rt =
   1.4 + 0.0i  -0.9 + 0.3i   1.6 + 0.0i   3.6 + 1.2i
   7.0 + 0.0i  -0.9 + 0.3i   1.6 + 0.0i  -2.7 - 0.9i
```

```
% "Equalization"
S_hatt = zeros(size(Rt));
for i=1:length(H)
    S_hatt(:,i) = Rt(:,i)/H(i);
end
S_hat = reshape(S_hatt.',1,N*n)
```

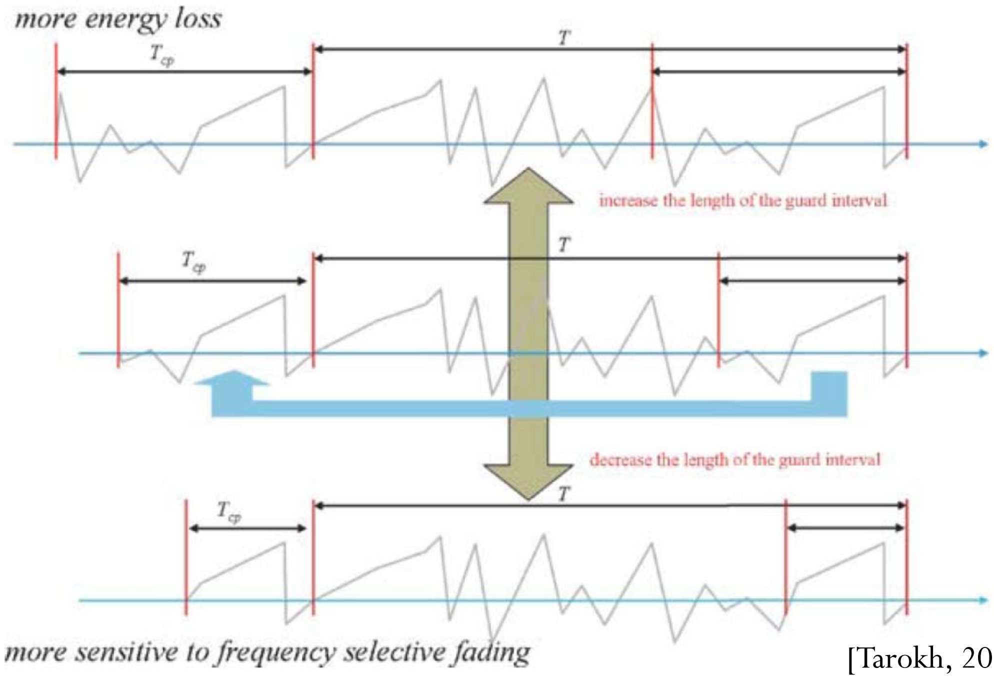
`% Divide Rk (the kth column) by Hk`

```
Shatt =
     1    -1     2     4
     5    -1     2    -3
```

```
S_hat = [1 -1 2 4 5 -1 2 -3]
```

OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.



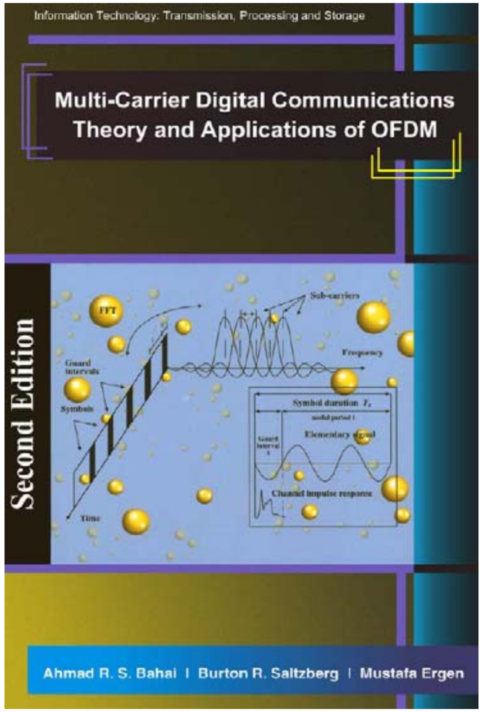
[Tarokh, 2009, Fig 2.9]

Summary

- The CP at the beginning of each block has two main functions.
- As guard interval, it prevents contamination of a block by ISI from the previous block.
- It makes the received block appear to be periodic of period N.
  - Turn regular convolution into circular convolution
  - Point-wise multiplication in the frequency domain

Reference

- A. Bahai, B. R. Saltzberg, and M. Ergen, ***Multi-Carrier Digital Communications: Theory and Applications of OFDM***, 2nd ed., New York: Springer Verlag, 2004.



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Multi-Carrier Digital Communications
Theory and Applications of OFDM

Second Edition

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